

ON NON-VOLTERRA QUADRATIC STOCHASTIC OPERATORS GENERATED BY A PRODUCT MEASURE

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Abstract

In this paper we describe a wide class of non-Volterra quadratic stochastic operators using N. Ganikhadjaev's construction of quadratic stochastic operators. By the construction these operators depend on a probability measure μ being defined on the set of all configurations which are given on a graph G . We show that if μ is the product of probability measures being defined on each maximal connected subgraphs of G then corresponding non-Volterra operator can be reduced to m number (where m is the number of maximal connected subgraphs of G) of Volterra operators defined on the maximal connected subgraphs. Our result allows to study a wide class of non-Volterra operators in the framework of the well known theory of Volterra quadratic stochastic operators.

Keywords: Quadratic stochastic operator, Volterra and Non-Volterra operators, simplex, product measure.

1 Introduction

It is well known that the principles of biological inheritance, initiated by Mendel allow an exact mathematical formulation. For this reason classical genetics can be regarded as a mathematical discipline. The fundamental investigations on these problems were carried out by S.N.Bernstein [2] and Reiersol [13].

The main mathematical apparatus of such investigations, to our knowledge, is the theory of quadratic stochastic operators. Such operator frequently arise in many models of mathematical genetics [1-11], [13], [15],[16].

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The quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{n-1} = \{x = (x_1, \dots, x_n) \in R^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$$

into itself, of the form

$$V : x'_k = \sum_{i,j=1}^n p_{ij,k} x_i x_j, \quad (k = 1, \dots, n), \quad (1)$$

where $p_{ij,k}$ are coefficients of heredity and

$$p_{ij,k} \geq 0, \quad \sum_{k=1}^n p_{ij,k} = 1, \quad (i, j, k = 1, \dots, n). \quad (2)$$

Note that each element $x \in S^{n-1}$ is a probability distribution on $E = \{1, \dots, n\}$.

The population evolves by starting from an arbitrary state (probability distribution on E) $x \in S^{n-1}$ then passing to the state Vx (in the next “generation”), then to the state V^2x , and so on.

For a given $x^{(0)} \in S^{n-1}$ the trajectory $\{x^{(l)}\}, l = 0, 1, 2, \dots$ of QSO (1) is defined by $x^{(l+1)} = V(x^{(l)})$, where $l = 0, 1, 2, \dots$. One of the main problems in mathematical biology consists in the study of the asymptotical behavior of the trajectories. This problem was fully solved [1-7] for Volterra QSO which is defined by (1), (2) and the additional assumption

$$p_{ij,k} = 0, \quad \text{if } k \notin \{i, j\}. \quad (3)$$

The biological treatment of condition (3) is rather clear: the offspring repeats the genotype of one of its parents.

In paper [6] the general form of Volterra QSO $V : x = (x_1, \dots, x_n) \in S^{n-1} \rightarrow V(x) = x' = (x'_1, \dots, x'_n) \in S^{n-1}$ is given:

$$x'_k = x_k \left(1 + \sum_{i=1}^n a_{ki} x_i \right), \quad (4)$$

where $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$.

In papers [6], [7] the theory of QSO (4) was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs (i.e. which do not satisfy the condition (3)) were not in completely studied. Because there is no any general theory which can be applied for investigation of non-Volterra operators. To the best of our knowledge, there are few papers devoted to such operators (see e.g. [1], [4]).

In papers [3],[5] a constructive description of QSOs is given. This construction depends on a probability measure μ and cardinality of a set of cells (configurations) which can be finite or continual. In this paper we describe QSOs using the construction of QSO for the general finite graph and probability measure μ which is product of measures defined on maximal subgraphs of the graph. We show that if μ is given by the product of the probability measures then corresponding non-Volterra operator can be reduced to m number (where m is the number of maximal

connected subgraphs) of Volterra operators defined on the maximal connected subgraphs. Then using the well known theory of Volterra operators we study the asymptotical behavior of the trajectories of the non-Volterra operators.

2 Construction of QSO

Each quadratic operator V can be uniquely defined by a cubic matrix $\mathbf{P} \equiv \mathbf{P}(V) = \{p_{ij,k}\}_{i,j,k=1}^n$ with condition (2). Usually [1], [2], [4], [6-11] the matrix \mathbf{P} is known. In [3], [5] a constructive description of \mathbf{P} is given.

Now we recall this construction.

Let $G = (\Lambda, L)$ be a finite graph without loops and multiple edges, where Λ is the set of vertexes and L is the set of edges of the graph.

Furthermore, let Φ be a finite set, called the set of alleles (in problems of statistical mechanics, Φ is called the range of spin). The function $\sigma : \Lambda \rightarrow \Phi$ is called a cell (in mechanics it is called configuration). Denote by Ω the set of all cells, this set corresponds to E in (1). Let $S(\Lambda, \Phi)$ be the set of all probability measures defined on the finite set Ω .

Let $\{\Lambda_i, i = 1, \dots, m\}$ be the set of maximal connected subgraphs (components) of the graph G . For a configuration $\sigma \in \Omega$ denote by $\sigma(M)$ its "projection" (or "restruction") to $M \subset \Lambda$: $\sigma(M) = \{\sigma(x)\}_{x \in M}$. Fix two cells $\sigma_1, \sigma_2 \in \Omega$, and put

$$\Omega(G, \sigma_1, \sigma_2) = \{\sigma \in \Omega : \sigma(\Lambda_i) = \sigma_1(\Lambda_i) \text{ or } \sigma(\Lambda_i) = \sigma_2(\Lambda_i) \text{ for all } i = 1, \dots, m\}.$$

Now let $\mu \in S(\Lambda, \Phi)$ be a probability measure defined on Ω such that $\mu(\sigma) > 0$ for any cell $\sigma \in \Omega$; i.e μ is a Gibbs measure with some potential [12], [14]. The heredity coefficients $p_{\sigma_1 \sigma_2, \sigma}$ are defined as

$$p_{\sigma_1 \sigma_2, \sigma} = \begin{cases} \frac{\mu(\sigma)}{\mu(\Omega(G, \sigma_1, \sigma_2))}, & \text{if } \sigma \in \Omega(G, \sigma_1, \sigma_2), \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Obviously, $p_{\sigma_1 \sigma_2, \sigma} \geq 0$, $p_{\sigma_1 \sigma_2, \sigma} = p_{\sigma_2 \sigma_1, \sigma}$ and $\sum_{\sigma \in \Omega} p_{\sigma_1 \sigma_2, \sigma} = 1$ for all $\sigma_1, \sigma_2 \in \Omega$.

The QSO $V \equiv V_\mu$ acting on the simplex $S(\Lambda, \Phi)$ and determined by coefficients (5) is defined as follows: for an arbitrary measure $\lambda \in S(\Lambda, \Phi)$, the measure $V(\lambda) = \lambda' \in S(\Lambda, \Phi)$ is defined by the equality

$$\lambda'(\sigma) = \sum_{\sigma_1, \sigma_2 \in \Omega} p_{\sigma_1 \sigma_2, \sigma} \lambda(\sigma_1) \lambda(\sigma_2) \quad (6)$$

for any cell $\sigma \in \Omega$.

The QSO construction is also closely related to the graph structure on the set Λ .

Theorem 1. [3] *The QSO (5) is Volterra if and only if the graph G is connected.*

3 Non-Volterra QSO

In this section we describe a condition on measure μ under which the QSO V_μ generated by μ can be studied using the theory of Volterra QSO.

Let $G = (\Lambda, L)$ be a finite graph and $\{\Lambda_i, i = 1, \dots, m\}$ the set of all maximal connected subgraphs of G . Denote by $\Omega_i = \Phi^{\Lambda_i}$ the set of all configurations defined on Λ_i , $i = 1, \dots, m$. Let μ_i be a probability measure defined on Ω_i , such that $\mu_i(\sigma) > 0$ for any $\sigma \in \Omega_i$, $i = 1, \dots, m$.

Consider probability measure μ on $\Omega = \Omega_1 \times \dots \times \Omega_m$ defined as

$$\mu(\sigma) = \prod_{i=1}^m \mu_i(\sigma_i), \quad (7)$$

where $\sigma = (\sigma_1, \dots, \sigma_m)$, with $\sigma_i \in \Omega_i, i = 1, \dots, m$.

By Theorem 1 if $m = 1$ then QSO constructed on G is Volterra QSO.

Theorem 2. *The QSO constructed by (5) with measure (7) is reducible to m separate Volterra QSOs.*

Proof. Take $\varphi = (\varphi_1, \dots, \varphi_m), \psi = (\psi_1, \dots, \psi_m) \in \Omega$. By construction

$$\Omega(G, \varphi, \psi) = \left\{ \sigma = (\sigma_1, \dots, \sigma_m) \in \Omega : \sigma_i \in \{\varphi_i, \psi_i\}, i = 1, \dots, m \right\}$$

and

$$p_{\varphi\psi,\sigma} = \begin{cases} \prod_{i=1}^m \frac{\mu_i(\sigma_i)}{\mu_i(\varphi_i) + \mu_i(\psi_i)}, & \text{if } \sigma \in \Omega(G, \varphi, \psi), \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

where we used the following equality

$$\mu(\Omega(G, \varphi, \psi)) = \sum_{\substack{\sigma_1, \dots, \sigma_m: \\ \sigma_i \in \{\varphi_i, \psi_i\}, i=1, \dots, m}} \prod_{i=1}^m \mu_i(\sigma_i) = \prod_{i=1}^m (\mu_i(\varphi_i) + \mu_i(\psi_i)).$$

Thus QSO (6) generated by measure (7) can be written as

$$\lambda'(\sigma) = \lambda'(\sigma_1, \dots, \sigma_m) = \sum_{\substack{\varphi=(\varphi_1, \dots, \varphi_m): \varphi_i \in \Omega_i \\ \psi=(\psi_1, \dots, \psi_m): \psi_i \in \Omega_i}} \prod_{i=1}^m \frac{\mu_i(\sigma_i) \mathbf{1}_{(\sigma_i \in \{\varphi_i, \psi_i\})}}{\mu_i(\varphi_i) + \mu_i(\psi_i)} \lambda(\varphi) \lambda(\psi). \quad (9)$$

Denote

$$X_{i,\omega} = \sum_{\substack{\sigma \in \Omega: \\ \sigma_i = \omega}} \lambda(\sigma) = \sum_{\substack{\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_m \\ \sigma_k \in \Omega_k, k \neq i}} \lambda(\sigma_1, \dots, \sigma_{i-1}, \omega, \sigma_{i+1}, \dots, \sigma_m). \quad (10)$$

From (9) we have

$$X'_{i,\omega} = \sum_{\substack{\sigma \in \Omega: \\ \sigma_i = \omega}} \lambda'(\sigma) = \sum_{\substack{\sigma \in \Omega: \\ \sigma_i = \omega}} \left[\sum_{\substack{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_m \\ \psi_1, \dots, \psi_m}} \frac{\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi_i)} \times \right. \\ \left. \prod_{\substack{j=1 \\ j \neq i}}^m \frac{\mu_j(\sigma_j) \mathbf{1}_{(\sigma_j \in \{\varphi_j, \psi_j\})}}{\mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\varphi_1, \dots, \varphi_{i-1}, \omega, \varphi_{i+1}, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_m) + \right.$$

$$\begin{aligned}
& \sum_{\substack{\varphi_1, \dots, \varphi_m \\ \psi_1, \dots, \psi_{i-1}, \psi_{i+1}, \dots, \psi_m}} \frac{\mu_i(\omega)}{\mu_i(\varphi_i) + \mu_i(\omega)} \times \\
& \prod_{\substack{j=1 \\ j \neq i}}^m \frac{\mu_j(\sigma_j) \mathbf{1}_{(\sigma_j \in \{\varphi_j, \psi_j\})}}{\mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\varphi_1, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_{i-1}, \omega, \psi_{i+1}, \dots, \psi_m) \Big] = \\
& 2 \sum_{\substack{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_m \\ \psi_1, \dots, \psi_m}} \frac{\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi_i)} \times \\
& \sum_{\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_m} \prod_{\substack{j=1 \\ j \neq i}}^m \frac{\mu_j(\sigma_j) \mathbf{1}_{(\sigma_j \in \{\varphi_j, \psi_j\})}}{\mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\varphi_1, \dots, \varphi_{i-1}, \omega, \varphi_{i+1}, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_m). \quad (11)
\end{aligned}$$

Note that

$$\sum_{\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_m} \prod_{\substack{j=1 \\ j \neq i}}^m \frac{\mu_j(\sigma_j) \mathbf{1}_{(\sigma_j \in \{\varphi_j, \psi_j\})}}{\mu_j(\varphi_j) + \mu_j(\psi_j)} = 1.$$

Thus from (11) we have

$$\begin{aligned}
& \text{RHS of (11)} = \\
& 2 \sum_{\substack{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_m \\ \psi_1, \dots, \psi_m}} \frac{\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi_i)} \lambda(\varphi_1, \dots, \varphi_{i-1}, \omega, \varphi_{i+1}, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_m) = \\
& \sum_{\substack{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_m \\ \psi_1, \dots, \psi_{i-1}, \psi_{i+1}, \dots, \psi_m}} \lambda(\varphi_1, \dots, \varphi_{i-1}, \omega, \varphi_{i+1}, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_{i-1}, \omega, \psi_{i+1}, \dots, \psi_m) + \\
& 2 \sum_{\psi_i \in \Omega_i \setminus \omega} \frac{\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi_i)} \sum_{\substack{\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_m \\ \psi_1, \dots, \psi_{i-1}, \psi_{i+1}, \dots, \psi_m}} \lambda(\varphi_1, \dots, \varphi_{i-1}, \omega, \varphi_{i+1}, \dots, \varphi_m) \lambda(\psi_1, \dots, \psi_m) = \\
& X_{i,\omega}^2 + \sum_{\psi \in \Omega_i \setminus \omega} \frac{2\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi)} X_{i,\omega} X_{i,\psi}.
\end{aligned}$$

Thus operator (9) can be rewritten as

$$X'_{i,\omega} = X_{i,\omega} \left(X_{i,\omega} + \sum_{\psi \in \Omega_i \setminus \omega} \frac{2\mu_i(\omega)}{\mu_i(\omega) + \mu_i(\psi)} X_{i,\psi} \right), \quad (12)$$

where $X_{i,\omega}$ is defined by (10), $\omega \in \Omega_i, i = 1, \dots, m$.

Note that $\sum_{\omega \in \Omega_i} X_{i,\omega} = 1$ for any $i = 1, \dots, m$. Using this equality from (12) we obtain

$$X'_{i,\omega} = X_{i,\omega} \left(1 + \sum_{\psi \in \Omega_i} \frac{\mu_i(\omega) - \mu_i(\psi)}{\mu_i(\omega) + \mu_i(\psi)} X_{i,\psi} \right). \quad (13)$$

Comparing this with (4) one can see that for each fixed i ($i = 1, \dots, m$) the operator (13) is Volterra operator $V^{(i)} : S^{|\Omega_i|-1} \rightarrow S^{|\Omega_i|-1}$. The theorem is proved.

Remark. The set Φ which we used for the set of spin values on G is the same for any configuration on $\Omega_i, i = 1, \dots, m$. Note that Theorem 2 can be easily extended for more general case: when each subgraph Λ_i has its own Φ_i .

4 The behavior of the trajectories

In this section using Volterra QSOs (13) we shall describe the behavior of trajectories of non-Volterra QSO (9).

Denote $a_{\varphi,\psi}^{(i)} = \frac{\mu_i(\omega) - \mu_i(\psi)}{\mu_i(\omega) + \mu_i(\psi)}$. It is easy to see that for each fixed $i \in \{1, \dots, m\}$ the coefficients $a_{\varphi,\psi}^{(i)}$ satisfy the following properties

$$a_{\varphi,\psi}^{(i)} = -a_{\psi,\varphi}^{(i)}, \quad |a_{\varphi,\psi}^{(i)}| \leq 1. \quad (14)$$

Thus QSO (13) has the same form with (4).

If for any $i \in \{1, \dots, m\}$ the asymptotical behavior of trajectories of QSO $V^{(i)}$ (i.e. (13)) is known, say $X_{i,\omega}^{(l)} \rightarrow X_{i,\omega}^*$, $l \rightarrow \infty$, then asymptotical behavior of V (i.e. (9)), say $\lambda^{(l)}(\sigma) \rightarrow \lambda^*(\sigma)$, $l \rightarrow \infty$, can be found from system of linear equations

$$\sum_{\sigma \in \Omega: \sigma_i = \omega} \lambda^*(\sigma) = X_{i,\omega}^*, \quad \omega \in \Omega_i, i = 1, \dots, m. \quad (15)$$

Now we recall known results ([6],[7]) about trajectories of (4) (i.e. (13) for fixed i).

Put

$$\partial S^{n-1} = \{x = (x_1, \dots, x_n) \in S^{n-1} : \prod_{j=1}^n x_j = 0\}, \quad \text{int} S^{n-1} = S^{n-1} \setminus \partial S^{n-1}.$$

Let $x^{(0)} \in S^{n-1}$ be the initial point. Denote by $\nu(x^{(0)})$ the set of limit points of the trajectory $\{x^{(l)}\}$ of the operator (4).

Theorem 3. [6]. 1) If $x^{(0)} \in \text{int} S^{n-1}$ is not a fixed point (i.e. $Vx^{(0)} \neq x^{(0)}$), then $\nu(x^{(0)}) \subset \partial S^{n-1}$.

2) The set $\nu(x^{(0)})$ either consists of a single point or is infinite.

3) If QSO (4) has an isolated fixed point $x^* \in \text{int} S^{n-1}$, then for any initial point $x^{(0)} \neq x^*$, the trajectory $\{x^{(l)}\}$ does not converge.

Corollary 4. The set of limit points $\tilde{\nu}(\lambda^0)$ of the QSO (9) has properties 1)-3) mentioned in Theorem 3.

Proof. 1) By Theorem 3 there is at least one $\tilde{\omega} \in \Omega_i$ with $X_{i,\tilde{\omega}}^* = 0$. Since $\lambda(\sigma) \geq 0$ from (15) we get that $\lambda^*(\sigma) = 0$ for all σ such that $\sigma_i = \tilde{\omega}$. This completes the proof of the property 1). Properties 2),3) also follow from (15).

Now we shall recall some notions from theory of tournaments (see [6]) for operator (4). Assume $a_{ki} \neq 0$ for $k \neq i$. Along with (4) consider the complete graph G_n with n vertices. Specify a direction on the edges of G_n as follows: the edge joining vertices k and i is directed from the k th to the i th vertex if $a_{ki} < 0$, and has the opposite direction if $a_{ki} > 0$. The directed graph thus obtained is called a tournament and denoted by T_n . A tournament is said to be strong if it is possible to go from any vertex to any other vertex with directions taken into account.

Further, after a suitable renumbering of the vertices of T_n we can assume that the subtournament T_r contains the first r vertices of T_n as its vertices. Obviously, $r \leq n$, and $r = n$ if and only if T_n is a strong tournament.

Theorem 5.[6]. *Suppose that T_n is not strong, and let $x^{(0)} \in \text{int}S^{n-1}$. If $j > r$, then $x_j^{(l)} \rightarrow 0$, at the rate of a geometric progression as $l \rightarrow \infty$.*

As corollary of Theorem 5 we have

Corollary 6. *Suppose that there is a $i_0 \in \{1, \dots, m\}$ such that the corresponding tournament $T^{(i_0)}$ of the operator $V^{(i_0)}$ is not strong, and assume $\lambda^0 \in \text{int}S^{n-1}$. Then there is a subset $\tilde{\Omega}_{i_0} \subset \Omega_{i_0}$ such that $\lambda^{(l)}(\sigma) \rightarrow 0$, for $\sigma \in \Omega$ with $\sigma_{i_0} \in \tilde{\Omega}_{i_0}$, at the rate of a geometric progression as $l \rightarrow \infty$.*

Example. Consider graph $G = (\Lambda, L)$ with $\Lambda = \{1, 2\}$ and $L = \emptyset$. Take $\Phi = \{A, a\}$. Then non-Volterra QSO (9) has the form

$$\begin{aligned} x'_1 &= x_1^2 + 2\beta_1 x_1 x_2 + 2\alpha_1 x_1 x_3 + 2\alpha_1 \beta_1 x_1 x_4 + 2\alpha_1 \beta_1 x_2 x_3 \\ x'_2 &= x_2^2 + 2\beta_2 x_1 x_2 + 2\alpha_1 \beta_2 x_2 x_3 + 2\alpha_1 x_2 x_4 + 2\alpha_1 \beta_2 x_1 x_4 \\ x'_3 &= x_3^2 + 2\alpha_2 x_1 x_3 + 2\alpha_2 \beta_1 x_2 x_3 + 2\beta_1 x_3 x_4 + 2\alpha_2 \beta_1 x_1 x_4 \\ x'_4 &= x_4^2 + 2\alpha_2 \beta_2 x_1 x_4 + 2\alpha_2 x_2 x_4 + 2\beta_2 x_3 x_4 + 2\alpha_2 \beta_2 x_2 x_3 \end{aligned} \tag{16}$$

where $\mu_1 = (\alpha_1, \alpha_2), \alpha_j \geq 0, j = 1, 2, \alpha_1 + \alpha_2 = 1; \mu_2 = (\beta_1, \beta_2), \beta_j \geq 0, j = 1, 2, \beta_1 + \beta_2 = 1$.

Putting $x_1 + x_2 = X_{1,1}$, $x_3 + x_4 = X_{1,2}$ and $x_1 + x_3 = X_{2,1}$, $x_2 + x_4 = X_{2,2}$ we get the Volterra operators like (13):

$$\begin{aligned} X'_{1,1} &= X_{1,1} \left(1 + (2\alpha_1 - 1)X_{1,2} \right) \\ X'_{1,2} &= X_{1,2} \left(1 + (2\alpha_2 - 1)X_{1,1} \right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} X'_{2,1} &= X_{2,1} \left(1 + (2\beta_1 - 1)X_{2,2} \right) \\ X'_{2,2} &= X_{2,2} \left(1 + (2\beta_2 - 1)X_{2,1} \right) \end{aligned} \tag{18}$$

For this example from Corollary 4 and 6 we obtain

Corollary 7. 1. *Any trajectory of non-Volterra QSO (16) has following limit*

$$\lim_{l \rightarrow \infty} x^{(l)} = \begin{cases} (1, 0, 0, 0), & \text{if } 2\alpha_1 > 1, 2\beta_1 > 1, \\ (0, 1, 0, 0), & \text{if } 2\alpha_1 > 1, 2\beta_1 < 1, \\ (0, 0, 1, 0), & \text{if } 2\alpha_1 < 1, 2\beta_1 > 1, \\ (0, 0, 0, 1), & \text{if } 2\alpha_1 < 1, 2\beta_1 < 1. \end{cases}$$

2. If $2\beta_1 = 1$ then $S_1 = \{x : x_3 = x_4 = 0\}$ and $S_2 = \{x : x_1 = x_2 = 0\}$ are the sets of fixed points for (16) and for any $x^{(0)} \notin S_1 \cup S_2$

$$\lim_{l \rightarrow \infty} x^{(l)} \in \begin{cases} S_1, & \text{if } 2\alpha_1 > 1, \\ S_2 & \text{if } 2\alpha_1 < 1. \end{cases}$$

3. If $2\alpha_1 = 1$ then $S_3 = \{x : x_2 = x_4 = 0\}$ and $S_4 = \{x : x_1 = x_3 = 0\}$ are the sets of fixed points for (16) and for any $x^{(0)} \notin S_3 \cup S_4$

$$\lim_{l \rightarrow \infty} x^{(l)} \in \begin{cases} S_3, & \text{if } 2\beta_1 > 1, \\ S_4 & \text{if } 2\beta_1 < 1. \end{cases}$$

4. If $2\alpha_1 = 2b_1 = 1$ then $S_5 = \{x : x_2 = x_4, x_1 = x_3\}$ and $S_6 = \{x : x_1 = x_2, x_3 = x_4\}$ are the sets of fixed points for (16).

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